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# Perturbation theory for infinite-component wave equations

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**Abstract.** A new covariant perturbation theory for infinite-component wave equations is constructed which is analogous to the Epstein–Waller and Lewis–Dalgarno method of atomic physics. Instead of infinite sums over discrete and continuous states the matrix elements involve Lie algebra or group elements and can be evaluated in closed form.

## 1. Introduction

The interest in further study of infinite-component wave equations has declined in recent years due to the existence of space-like solutions, and due to the difficulties of constructing a fully-fledged theory with them. We proposed a solution to the first problem in our previous paper (Barut and Nagel 1977), where we show the physical interpretation and significance of the space-like solutions. The second problem can now be viewed as a field theory of composite particles. Such a field theory will undoubtedly have different features than the usual local field theory. Short of the goal of such a complete field theory, external field problems can be treated by a perturbation expansion. We develop here a covariant perturbation theory for a class of covariant infinite-component wave equations. It is different from a perturbation theory of the mass or the Hamiltonian operator, but amounts essentially to a perturbation of the equation for the principal quantum number. In an infinite-component field theory, for second-order and higher, matrix elements usually extend over all the infinitely many discrete and continuous states. However, by using the underlying Lie algebra structure and performing a perturbation theory on the so called 'group states' we can avoid these infinite sums and reduce the matrix elements to those of Lie algebra or group elements or their products which can be evaluated in closed form. An analogous situation occurs in atomic physics, where in the calculation of the Stark effect in H atom a perturbation theory is performed not on the Hamiltonian, but on a suitably chosen function of the Hamiltonian, for which the present paper provides a group theoretical explanation (see references in Schiff 1955; also see Waller 1926, Epstein 1926, Dalgarno and Lewis 1955).

We take as the prototype (and the most important case) of the unperturbed equation the form

$$(J_\mu P^\mu + \beta S + \gamma)\tilde{\Psi}(P) = 0, \tag{1}$$

where

$$J_\mu = \alpha_1 \Gamma_\mu + \alpha_2 P_\mu + \alpha_3 P_\mu S + i\alpha_4 L_{\mu\nu} q^\nu. \tag{2}$$

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Here $\Gamma_\mu$, $S$, $L_{\mu\nu}$ are the generators of the dynamical group SO(4, 2) and $\alpha_i$, $\beta$, $\gamma$ are constants of the theory. Our considerations will also automatically apply to other infinite-component wave equations. We shall write (1) also as

$$\Omega\tilde{\Psi}(P)=0. \tag{1'}$$

The complete set of solutions of (1) is known, which includes also solutions with space-like momenta (Barut 1973).

Let the interaction be denoted by $W$. For example, we can perturb equation (1) via the minimal coupling to the external field: $P_\mu \to P_\mu - eA_\mu$, and obtain the new equation in momentum space:

$$(J_\mu P'^\mu + \beta S + \gamma + \tilde{W})\tilde{\Psi}'(P')=0. \tag{3}$$

Because the mass spectrum of the new equation (3) is different from (1) we have denoted the momentum by $P'_\mu$. Thus $P_\mu P^\mu = M^2$, but $P'_\mu P'^\mu \neq M^2$. The interaction operator in this case is

$$\tilde{W} = eJ_\mu A^\mu + e^2(\alpha_2 + \alpha_3 S)A_\mu A^\mu.$$

We do not yet have complete Feynman rules for calculating the $S$ matrix elements with infinite-component fields. However, equation (3) has well defined solutions as $e \to 0$, hence an obvious approach in this case of external fields is to consider the interaction terms as small perturbations. Thus we attempt to solve equation (3) in a perturbation series.

As a first step we transform (3) formally to the rest frame of the system in the presence of external fields: $P'_\mu = (M', 0, 0, 0)$:

$$(J_0 M' + \beta S + \gamma + \tilde{W})\tilde{\Psi}'(0)=0. \tag{4}$$

Note that the interaction terms are Lorentz scalars.

In the second step we eliminate $S \equiv \Gamma_4 = L_{46}$ by performing the usual 'tilt' operation (Barut 1973). This is a transformation, similar to Foldy–Wouthuysen transformation but infinite-dimensional, whose purpose is to diagonalize the equation. For this purpose we set

$$\begin{aligned} \Psi'(0) &= \mathrm{e}^{-\mathrm{i}\theta' T}\tilde{\Psi}'(0) \\ W &\equiv \mathrm{e}^{-\mathrm{i}\theta' T}\tilde{W}\,\mathrm{e}^{\mathrm{i}\theta' T} \end{aligned} \tag{5}$$

where $T \equiv L_{45}$ is the dilation operator in SO(4, 2). By proper choice of $\theta'$ we can make the coefficient of $S$ in (4) zero and achieve a diagonalization:

$$(a'\Gamma_0 + b' + W)\Psi'(0)=0, \tag{6a}$$

where

$$\begin{aligned} a' &= [\alpha_1^2 M'^2 - (\alpha_3 M'^2 + \beta)^2]^{1/2}[-\mathrm{sgn}(\alpha_2 M'^2 + \gamma)] \\ b' &= \alpha_2 M'^2 + \gamma \end{aligned} \tag{6b}$$

and

$$\tanh\theta' = -\frac{(\alpha_3 M'^2 + \beta)}{\alpha_1 M'}. \tag{7}$$

## 2. Perturbation expansions

In (6*a*) $\Gamma_0$ is a fixed operator with a well defined discrete spectrum determined by the representation of SO(4, 2). We perform a perturbation expansion on the coefficients *a*, *b*, as well as on *W* and on the wavefunction $\Psi'(0)$:

$$a' = a + \lambda a_1 + \lambda^2 a_2 + \ldots \tag{8a}$$

$$b' = b + \lambda b_1 + \lambda^2 b_2 + \ldots \tag{8b}$$

$$W = \lambda W_1 + \lambda^2 W_2 + \ldots \tag{8c}$$

$$\Psi'(0) = \Psi + \lambda \Psi_1 + \ldots . \tag{8d}$$

These expansions of *a*, *b*, and *W* result from the expansion of $M'$, hence $\theta'$:

$$M' = M + \lambda M_1 + \lambda^2 M_2 + \ldots$$

$$\theta' = \theta + \lambda \theta_1 + \ldots .$$

Then, from (6*a*), we obtain for various powers of $\lambda$ the equations

$$(a\Gamma_0 + b)\Psi = 0 \tag{9}$$

$$(a_1\Gamma_0 + b_1 + W_1)\Psi_0 + (a\Gamma_0 + b)\Psi_1 = 0 \tag{10}$$

$$(a_2\Gamma_0 + b_2 + W_2)\Psi_0 + (a_1\Gamma_0 + b_1 + W_1)\Psi_1 + (a\Gamma_0 + b)\Psi_2 = 0. \tag{11}$$

Now $\Psi'(0)$ in (6*a*) are chosen as eigenstates of $\Gamma_0$:

$$\Gamma_0|n\rangle = n|n\rangle, \tag{12}$$

which have been called group states. The inner product is just $(\Psi_1, \Psi_2)$ or $\langle n_1|n_2\rangle$. We take the inner product of (9)–(11) from the left with $\langle n|$, $\langle n|n\rangle = 1$:

$$an + b = 0 \tag{13}$$

$$a_1 n + b_1 + \langle n|W_1|n\rangle + (an + b)\langle n|\Psi_1\rangle = 0 \tag{14}$$

$$a_2 n + b_2 + \langle n|W_2|n\rangle + (a_1 n + b_1)\langle n|\Psi_1\rangle + \langle n|W_1|\Psi_1\rangle + (an + b)\langle n|\Psi_2\rangle = 0. \tag{15}$$

Equation (13) is the unperturbed equation and, with the values of *a* and *b* from (6*a*) (i.e. $M' = M$), gives the spectrum equation

$$\Omega(M, \alpha_i, \beta, \gamma, n) = 0, \tag{16}$$

which we solve for *m*.

In equations (14) and (15) we assume, as usual in perturbation theory, that

$$\langle n|\Psi_1\rangle = 0, \qquad \langle n|\Psi_2\rangle = 0. \tag{17}$$

Hence

$$a_1 n + b_1 = -\langle n|W_1|n\rangle \equiv -\gamma_1 \tag{18}$$

and

$$a_2 n + b_2 = -\langle n|W_2|n\rangle - \langle n|W_1|\Psi_1\rangle. \tag{19}$$

Now the operator that projects out the state $|n\rangle$ is $P \equiv 1 - |n\rangle\langle n|$ which commutes with $(a_i\Gamma_0 + b_i)$ but not necessarily with $W_1$. Applying *P* to equation (10) we get

$$(a_1\Gamma_0 + b_1)P|n\rangle + PW_1|n\rangle + (a\Gamma_0 + b)P\Psi_1 = 0,$$

or, since $P|n\rangle = 0$, $P\Psi_1 = \Psi_1$ by (17),

$$\Psi_1 = -(a\Gamma_0 + b)^{-1} P W_1 |n\rangle. \tag{20}$$

We insert this in (19):

$$a_2 n + b_2 = -\langle n|W_2|n\rangle + \langle n|W_1(a\Gamma_0 + b)^{-1} P W_1|n\rangle \equiv -\gamma_2 - \gamma_3. \tag{21}$$

We now add (13), (18) and (21) together:

$$(a + a_1 + a_2 + \ldots)n + (b + b_1 + b_2 + \ldots) + \gamma_1 + \gamma_2 + \gamma_3 = 0,$$

or

$$a'n + b' + \gamma_1 + \gamma_2 + \gamma_3 = 0. \tag{22}$$

If we insert $a'$ and $b'$ from (6) into (22) we obtain

$$[-\mathrm{sgn}(\alpha_2 M'^2 + \gamma)][\alpha_1^2 M'^2 - (\alpha_3 M'^2 + \beta)^2]^{1/2} n + \gamma + \alpha_2 M'^2 + \gamma_1 + \gamma_2 + \gamma_3 = 0. \tag{23}$$

This would be identical to the unperturbed equation if $\gamma_1 = \gamma_2 = \gamma_3 = 0$, so the effect of the perturbation is to change the parameter $\gamma$, which is actually clear from equation (4). Solving equation (23) for $M'$, we get to second order

$$M' = M - \frac{\gamma_1}{N_n^2} - \frac{(\gamma_2 + \gamma_3)}{N_n^2} - \frac{\gamma_1^2}{2N_n^4}\left(1 + \frac{\alpha_1^2(\alpha_1^2 - 4\beta\alpha_3)n^2 M^2}{(\alpha_2 M^2 + \gamma)^2 Q}\right) \tag{24}$$

where

$$N_n^2 \equiv \langle n|J_0|n\rangle = (n\alpha_1 \cosh\theta_n + 2\alpha_2 M_n + 2n\alpha_3 M_n \sinh\theta_n),$$

and

$$Q \equiv \left[\left(\alpha_1^2 - 2\beta\alpha_3 - \frac{2\gamma\alpha_2}{n^2}\right)^2 - 4\left(\alpha_3^2 + \frac{\alpha_2^2}{n^2}\right)\left(\beta^2 + \frac{\gamma^2}{n^2}\right)\right]^{1/2}$$

with $\theta_n$ given as in (7). The numbers $\gamma_1$, $\gamma_2$, $\gamma_3$ have been defined in (18) and (21), hence for any state $|n\rangle$ the shifted mass $M'$ can be evaluated using formulae (4), (5), (8*c*), (18), (21) and (24) and the perturbation theory is complete.

It is instructive to compare the present perturbation theory based on equation (6*a*), and the results of equations (18) and (21), with the usual second-order expression (equations (3) or (4)):

$$\sum_n \frac{\langle\tilde{\Psi}'|\tilde{W}|\tilde{\Psi}'_n\rangle\langle\tilde{\Psi}'_n|\tilde{W}|\tilde{\Psi}'\rangle}{E_n - E_0}. \tag{25}$$

Here the states $\tilde{\Psi}'$ are the physical states. With (5), (25) can be written as

$$\sum_n \frac{\langle\Psi|e^{-i\theta T}\tilde{W}\, e^{i\theta_n T}|n\rangle\langle n|e^{-i\theta_n T}\tilde{W}\, e^{i\theta T}|\Psi\rangle}{E_n - E_0}.$$

In the operator $\exp(i\theta_n T)$, $\theta_n$ is different for each $n$, which necessitates the evaluation, in the infinite sum, of each term separately. In contrast, equation (21) can be evaluated in closed form.

The Lorentz transformation of (3) into (4) is, in the presence of external fields, at first a formal operation as we have noted. The composite system in an external field will undergo a complex motion consisting of a centre of mass motion and an internal

deformation. We are interested in the *intrinsic* properties of the systems described by our equation, like magnetic moment and electromagnetic polarizabilities, and not in the motion of the centre of mass. For this purpose we evaluate the Lorentz transformation of the interaction

$$e^{-i\xi.M} W e^{i\xi.M} \tag{26}$$

at $\boldsymbol{P} = 0$. The coordinate $\boldsymbol{X}$ in the interaction $W$ transforms under (26) as

$$\boldsymbol{X} \to \boldsymbol{X} + \frac{\boldsymbol{M}}{M'} + \boldsymbol{P}\text{-dependent terms.}$$

The first term refers to the centre-of-mass motion and the second term to the internal or spin space. In the non-relativistic limit this procedure corresponds exactly to separating centre of mass and relative coordinates. At $\boldsymbol{P} = 0$ we can treat to lowest order $M'$ as a number and then use our perturbation formulae, thereby obtaining finite formulae for energy shifts of the composite system or particle with spin.

The application of this theory to the calculation of electromagnetic polarizabilities is discussed elsewhere (Barut and Nagel 1976).

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